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MULTIPLE RECURRENCE AND INFINITE MEASURE PRESERVING ODOMETERS

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ABSTRACT

A family of infinite measure preserving odometers is presented which exhibit examples of p-recurrent but not $p+1$-recurrent ergodic transformations for every $p > 1$.

Introduction

Furstenberg, using ergodic theory, gave a new proof of Szemerédi's Theorem [15], which says that every sequence of integers of positive Banach density contains arbitrarily long arithmetic sequences. He did this by showing that Szemerédi's theorem was equivalent to Furstenberg's multiple recurrence Theorem; see [7] where he proved that every finite measure preserving transformation is multiply

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recurrent. However, we show that for infinite measure spaces the situation is different. It is possible for an ergodic measure preserving transformation on an infinite measure space to be multiply recurrent or not. Furthermore, for every $p > 1$ we will demonstrate the existence of ergodic, infinite measure preserving transformations which are $(p-1)$-recurrent but not p-recurrent.

The examples we construct will be members of a class of infinite measure preserving odometers. This type of construction has been studied by many authors ([1], [2], [9], [10], [12], [11], [13], [14]). In general a nonsingular product measure is given for the odometer and it is then proven that there exists an equivalent infinite invariant measure. This is done by analyzing either the ratio set (for example see [2]) or by showing the Radon–Nikodym derivative is 1 on an induced set (for example, see [1] and [13]).

The odometers

In this section we present the basic notation and construction of the transformations. It is easy to view these as rank-one transformations built by a cutting and stacking method as in Friedman [6]; however, we prefer to present them as odometers. Each odometer T will depend upon a sequence $\mathcal{M}$ of positive integers, $\mathcal{M} = \{m_i \geq 2 |\ i \geq 0\}$, and will be ergodic and infinite measure preserving. We exhibit a close connection between the multiply recurrent feature of T and the boundedness of a subsequence of $\mathcal{M}$; see the Theorem below. This explains, with respect to multiply recurrence, the unusual behavior of these odometers.

Let $\mathcal{M} = \{m_i \geq 2 |\ i \geq 0\}$ be a set of integers, and for each $i = 0, 1, 2, \ldots$ let $(\eta_{i,0}, \eta_{i,1}, \ldots, \eta_{i,m_i-1})$ be a set of m_i positive numbers with $\sum_{0 \leq j < m_i} \eta_{i,j} = 1$. In this article we choose the numbers $\eta_{i,j}$ such that if i is an even integer then $\eta_{i,j} = 1/m_i$ for all $0 \leq j < m_i$, and if i is an odd integer then $\eta_{i,0} = \eta_i$ for some η_i with $0 < \eta_i < 1$, and $\eta_{i,j} = (1 - \eta_i)/(m_i - 1)$ for $0 < j < m_i$; we also require that the numbers $\eta_{2k+1} > 0$ for $k \geq 0$ are such that $\prod_{k \geq 0} \eta_{2k+1} = P > 0$. For each $i \geq 0$ we consider the finite measure space $(\mathbf{Z}_i, \mathcal{B}_i, \mu_i)$, where $\mathbf{Z}_i$ consists of the m_i points $\{0, 1 \ldots, m_i - 1\}$, $\mathcal{B}_i$ is the set of all subsets of $\mathbf{Z}_i$, and the measure μ_i is defined by $\mu_i(j) = \eta_{i,j}$ for $0 \leq j < m_i$. It follows that $(\mathbf{Z}, \mathcal{B}, \mu) = \prod_{i \geq 0} (\mathbf{Z}_i, \mathcal{B}_i, \mu_i)$ is a finite measure space with $\mu(\mathbf{Z}) = 1$.

We eliminate from the set $\mathbf{Z}$ the countable set of points

$$\bigcup_{k \geq 0} \{(x_0, x_1, , \ldots) \in \mathbf{Z} |\ x_i = 0 \text{ for all } i \geq k\}$$

and

$$\bigcup_{k\geq 0}\{(x_0,x_1,,\ldots)\in \mathbf{Z}|\ x_i=m_i-1 \text{ for all } i\geq k\},$$

and denote the resulting space by $\mathbf{Z}$ again. For a point $x=(x_0,x_1,\ldots)\in\mathbf{Z}$ and for $i\geq 0$, we shall say that x_i is the i-th coordinate of x. We let T be the usual dyadic adding machine defined on $\mathbf{Z}$. Namely, we let the operation $+$ in $\mathbf{Z}$ be addition (mod m_i) in each i-th coordinate together with the appropriate carry, and for $x\in\mathbf{Z}$ we define $Tx=x+e$ where $e=(1,0,0,\ldots)$. It follows that the transformation T is a 1-1 non-singular transformation defined on the finite measure space $(\mathbf{Z},\mathcal{B},\mu)$.

Next we let $X_0=\{(x_0,x_1,\ldots)\in\mathbf{Z}|\ x_{2j+1}=0 \text{ for all } j\geq 0\}$ and let $\mathbf{X}=\bigcup_{i\geq 0}T^iX_0$. It is clear then that $\mathbf{X}$ is an invariant set under the transformation T. We also note that $\mu(X_0)=P\ (=\prod_{k\geq 0}\eta_{2k+1})$ and $\mu(\mathbf{X})=1$. In other words, the measure μ lives on the invariant space $\mathbf{X}$. Therefore, we abandon the space $\mathbf{Z}$ and consider the non-singular transformation T to be defined on the finite measure space $(\mathbf{X},\mathcal{B},\mu)$, where $\mathcal{B}$ and μ are the measurable sets and the measure induced on the set $\mathbf{X}$, respectively.

The following Proposition 1 follows from the work of a number of authors as stated above.

PROPOSITION 1: *The transformation T is ergodic on $(\mathbf{X},\mathcal{B},\mu)$, and there exists an infinite invariant measure m defined on $(\mathbf{X},\mathcal{B})$ which is equivalent with the measure μ.*

For a given sequence of integers $\mathcal{M}=\{m_i\geq 2\}$ we shall refer to the ergodic measure preserving transformation T constructed above on the infinite measure measure space $(\mathbf{X},\mathcal{B},m)$ as an odometer associated with the sequence $\mathcal{M}$. When $m_i=2$ for all $i\geq 0$ then the transformation T is isomorphic to the one constructed in [8].

Properties of the sequence $\mathcal{M}$ and the transformation T

The sequence $\mathcal{M}=\{m_i\geq 2\}$ of integers that was used to define the measure space $(\mathbf{X},\mathcal{B},m)$ and the odometer T associated with it possesses some number theoretic properties. The sequence $\mathcal{M}$ also imposes some important geometric restrictions on the transformation T.

Let us define the sequence $\{M_i|\ i\geq 0\}$ by $M_0=1$ and $M_i=m_{i-1}M_{i-1}$ for $i>0$. We let $\mathbf{N}=\{0,1,2,\ldots\}$; it follows that every integer $n\in\mathbf{N}$ has a unique representation as a finite sum of integers of the form n_iM_i with $0\leq n_i<m_i$. In

other words, $n = \sum_{0 \leq i < \infty} n_i M_i$, such that for some $k = k(n) > 0$ we have $n_i = 0$ for all $i > k$, and $0 \leq n_i < m_i$ otherwise. We shall write this representation of $n \in \mathbf{N}$ as $n = (n_0, n_1, \ldots)$, and call it the $\mathcal{M}$-adic representation of n, and refer to n_i as the i-th coordinate of n for $i \geq 0$. We shall denote by $\text{ord}_{\mathcal{M}}(n)$ the smallest non-negative integer i for which the i-th coordinate of n is greater than 0; for the integer 0 we shall say that $\text{ord}_{\mathcal{M}}(0) = \infty$.

For a subset $\mathcal{A} \subset \mathbf{N}$ let us denote $\mathcal{A} - \mathcal{A} = \{n \in \mathbf{N} | \, n = a - a' \text{ for } a, a' \in \mathcal{A}\}$, and let $\mathbf{IP}\{\mathcal{A}\}$ be the set of all integers that are sums of finite subsets of $\mathcal{A}$. In this note we shall assume that 0 belongs to every **IP** set.

In $\mathbf{N}$ let us consider the two subsets $\mathcal{E} = \mathbf{IP}(\bigcup_{k \geq 0} \{m_{2k} - 1 \text{ copies of } M_{2k}\})$ and $\mathcal{F} = \mathbf{IP}(\bigcup_{k \geq 0} \{m_{2k+1} - 1 \text{ copies of } M_{2k+1}\})$. It follows that $\mathcal{E} \oplus \mathcal{F} = \mathbf{N}$, where by $\mathcal{E} \oplus \mathcal{F}$ we mean $\{e + f | \, e \in \mathcal{E}, \, f \in \mathcal{F}\}$ such that if $e + f = e' + f'$ then $e = e'$ and $f = f'$; see [3]. We note that $\text{ord}_{\mathcal{M}}(n)$ is an even integer for every $n \in \mathcal{E} - \mathcal{E}$, and $\text{ord}_{\mathcal{M}}(n)$ is an odd integer for every $n \in \mathcal{F} - \mathcal{F}$. Thus, the sequences $\mathcal{E}$ and $\mathcal{F}$ (the odd and even parts of $\mathcal{M}$) exhibit complementing or dual properties as subsequences of $\mathbf{N}$. In a similar way, we show that these sequences exhibit dual geometric (wandering and recurrent) properties of the transformation T.

We state the following Proposition without proof. The proof is straightforward; see [5].

PROPOSITION 2: *Let $(\mathbf{X}, \mathcal{B}, m)$ be the σ-finite measure space and T the ergodic measure preserving odometer associated with the sequence $\mathcal{M} = \{m_i \geq 2\}$. Then for the subset $X_0 \subset \mathbf{X}$ and the set of integers $\mathcal{F}$ described above we have $\mathbf{X} = \bigcup_{f \in \mathcal{F}} T^f X_0$.*

Another way of stating Proposition 2 is: *the set X_0 is an exhausting weakly wandering set for the odometer transformation T under the sequence $\mathcal{F}$* (see [5]). We note that the sequence $\mathcal{F}$ is related to the 'odd-indexed' subsequence of $\mathcal{M}$. In the next section we consider the 'even-indexed' subsequence of $\mathcal{M}$.

The sequence $\mathcal{M}$ and multiple recurrence

Definition: Let $p > 0$ be a positive integer. The transformation T defined on the measure space $(\mathbf{X}, \mathcal{B}, m)$ is said to be ***p*-recurrent** if for any measurable set $B \in \mathcal{B}$ with $m(B) > 0$ there exists a positive integer $n > 0$ such that $m(B \cap T^n B \cap \cdots \cap T^{pn} B) > 0$. The transformation T is said to be **multiply recurrent** if T is p-recurrent for every $p > 0$.

We need to consider $2k$-rectangles in the space $\mathbf{X}$. For $2k > 0$ let $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{2k-1})$ be a set of $2k$ integers, such that $0 \leq \epsilon_j < m_j$ for $0 \leq j < 2k$.

Then a $2k$-**rectangle** in $\mathbf{X}$ is the subset

$$[\epsilon_0, \epsilon_1, \ldots, \epsilon_{2k-1}] = \{(x_0, x_1, \ldots) | x_j = \epsilon_j \text{ for } 0 \le j < 2k, \text{ and } x_{2i+1} = 0, \forall i \ge k\}.$$

The $2k$-rectangles in $\mathbf{X}$ approximate the subsets of $\mathbf{X}$ of finite measure.

THEOREM: *Let T be the ergodic measure preserving odometer on $(\mathbf{X}, \mathcal{B}, m)$ associated with the sequence $\mathcal{M} = \{m_i \ge 2\}$. Then*

(i) $\overline{\lim}_{i\to\infty} m_{2i} = \infty$ *if and only if T is multiply recurrent.*

(ii) *If $\overline{\lim}_{i\to\infty} m_{2i} = p < \infty$, then T is $(p-1)$-recurrent but not p-recurrent.*

LEMMA 1: *For $k > 0$ let R be the union of $2k$-rectangles in $\mathbf{X}$. Then for any integer j with $0 < j < m_{2k}$ we have $m(R \cap T^{M_{2k}} R \cap \cdots \cap T^{jM_{2k}} R) = (1 - j/m_{2k}) m(R)$.*

Proof: Let R be the union of $2k$-rectangles in $\mathbf{X}$ for some $k > 0$. Then any $2k$-rectangle $[\epsilon_0, \ldots, \epsilon_{2k-1}]$ in $\mathbf{X}$ splits into m_{2k} $(2k+2)$-rectangles; namely, $[\epsilon_0, \ldots, \epsilon_{2k-1}, \eta, 0]$ for $0 \le \eta < m_{2k}$. Each one of these $(2k+2)$-rectangles has measure $(1/m_{2k}) m([\epsilon_0, \ldots, \epsilon_{2k-1}])$, and

$$T^{M_{2k}}[\epsilon_0, \ldots, \epsilon_{2k-1}, \eta, 0] = [\epsilon_0, \ldots, \epsilon_{2k-1}, \eta + 1, 0] \quad \text{for } 0 \le \eta < m_{2k} - 1.$$

Also, the set $T^{M_{2k}}[\epsilon_0, \ldots, \epsilon_{2k-1}, m_{2k} - 1, 0] = [\epsilon_0, \ldots, \epsilon_{2k-1}, 0, 1]$ is disjoint from any $2k$-rectangle in $\mathbf{X}$. ∎

LEMMA 2: *Let $x \in X_0$, and let $n > 0$ be a positive integer with $\mathrm{ord}_{\mathcal{M}}(n) = r$. Then there exists an integer j with $1 \le j \le m_r$ such that $T^{jn}x \notin X_0$.*

Proof: Let $x \in X_0$, $n > 0$, and $\mathrm{ord}_{\mathcal{M}}(n) = r$. In the $\mathcal{M}$-adic representation of the integer n let us denote by s the r-th coordinate of n. We note that $0 < s < m_r$, and for all $i < r$ the i-th coordinate of n is 0.

Suppose $\mathrm{ord}_{\mathcal{M}}(n) = r$ is an odd integer; then the r-th coordinate of $T^n x$ will equal $s > 0$. This says that $T^n x \notin X_0$. Thus $j = 1$ in this case.

Therefore, we assume that $\mathrm{ord}_{\mathcal{M}}(n) = r$ is an even integer. Let t equal the r-th coordinate of the point $x \in X_0$. It is clear that $0 \le t < m_r$, and the $(r+1)$-th coordinate of x equals 0. Let $q =$ the $(r+1)$-th coordinate of n; it follows that $0 \le q < m_{r+1}$.

If $0 < q < m_{r+1} - 1$, then since the $(r+1)$-th coordinate of $T^n x$ is > 0, it follows that $T^n x \notin X_0$, and thus $j = 1$.

Next we let $q = 0$ and let j be the smallest positive integer such that $t + js \ge m_r$. This will insure a carry to the (r+1)-th coordinate of the $T^{jn}(x)$. It follows that $T^{jn} \notin X_0$, and $1 \le j \le m_r$.

Finally, we let $q = m_{r+1} - 1$. If $t_1 + s < m_r$ then $T^{jn}x \notin X_0$ for $j = 1$. Otherwise, we let $t_2 = t_1 + s - m_r$ and note that $0 \leq t_2 < t_1$. It follows that if $t_2 + s < m_r$ then $T^{jn}x \notin X_0$ for $j = 2$. Otherwise, we let $t_3 = t_2 + s - m_r$ and note that $0 \leq t_3 < t_2$. We continue this way for at most m_r steps and stop when $t_j + s < m_r$ for some j with $1 \leq j \leq m_r$. ∎

For the proof of the Theorem we need the following Corollary. It is a sharpened version of Lemma 2, and is proven in a similar way. We omit its proof.

COROLLARY: *For some integer $q > 0$ let*

$$V = \{(x_0, x_1, \ldots) \in X_0 | x_i = 0 \textit{ for } i \leq q\}.$$

Let $x \in V$, and let $n > 0$ be a positive integer with $\mathrm{ord}_{\mathcal{M}}(n) = r$. Then there exists an integer j with $1 \leq j \leq m_r$ such that $T^{jn}x \notin V$.

Proof of Theorem: Suppose $\overline{\lim}_{i\to\infty} m_{2i} = \infty$, and let $B \in \mathcal{B}$ be a set of positive measure. Then for any integer $p > 0$ there exist arbitrarily large even integers $2k$ such that $1 - p/m_{2k} \geq 1/p$. By possibly considering a subset of B, we may assume that $0 < m(B) < \infty$. Since the $2k$-rectangles approximate the sets of finite measure, it follows that there exists an even integer $2k > 0$ and a set R, which is the union of $2k$-rectangles in $\mathbf{X}$, such that $m(R \Delta B) < (1/p)m(R)$. Lemma 1 then implies

$$m(R \cap T^{M_{2k}} R \cap \cdots \cap T^{(p-1)M_{2k}} R) = (1 - p/m_{2k}) m(R) > (1/p) m(R).$$

It follows that $m(B \cap T^{M_{2k}} B \cap \cdots \cap T^{(p-1)M_{2k}} B) > 0$, and this shows that T is $(p-1)$-recurrent for any $p > 0$.

Next we let $p = \overline{\lim}_{i\to\infty} m_{2i} < \infty$. Then there exists an integer $q > 0$, and we assume it to be even, such that $m_{2i} \leq p$ for all $2i > q$. From the corollary to Lemma 2 when $\mathrm{ord}_{\mathcal{M}}(n) \geq q$, it follows that if

$$V = \{(x_0, x_1, \ldots) \in X_0 | x_i = 0 \text{ for } 0 \leq i \leq q\} \quad \text{and} \quad x \in V,$$

then there exists an integer j with $1 \leq j \leq p$ such that $T^{jn}x \notin V$. This implies that $V \cap T^n V \cap \cdots \cap T^{pn} V = \emptyset$, which says that the transformation T is not p-recurrent. In particular, T is not multiply recurrent.

Finally we show that the transformation T is $(p-1)$-recurrent in this case. We note that $p \geq 2$. Since $p = \overline{\lim}_{i\to\infty} m_{2i+1} < \infty$ it follows that there exist arbitrarily large integers $2k > 0$ such that $m_{2k} = p$. Now let $B \in \mathcal{B}$ be a set of positive measure. Again we may assume that $0 < m(B) < \infty$, and choose a large integer $2k > 0$ and a set R, which is the union of $2k$-rectangles in $\mathbf{X}$, such that $m(R\Delta B) < (1/p)m(R)$. We let $M_{2k} = \prod_{1\le j\le 2k} m_j$; Lemma 1 then implies $m(R \cap T^{M_i}R \cap \cdots \cap T^{(p-1)M_i}R) = (1/p)m(R)$. It follows that

$$m(B \cap T^{M_i}B \cap \cdots \cap T^{(p-1)M_i}B) > 0,$$

and this shows that T is $(p-1)$-recurrent. ∎

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